Some consistency issues in multi-criteria decision making

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Abstract: The complexity of decision-making problems included under the multi-criteria decision-making (MCDM) paradigm has favored the proliferation of many schools of thought and varied methodologies. It has not yet been possible to prove the supremacy of any of these approaches. Moreover, in some cases it is difficult to combine the theoretical validity of the approximations with their practical appropriateness. It seems that rigor and applicability are two opposing concepts, something that should not be so. It is our responsibility to bridge the gap. To reduce the gap between theory and practice and use effective methodological approaches, it is necessary to combine the rigor and objectivity of traditional science with the realism and subjectivity of human behavior. The analytic hierarchy process (AHP) perfectly combines a classical axiomatic foundation, and offers the objectivity of the traditional scientific method, with an excellent adaptation to the real behavior of individuals and systems in decision making (which connects with behavioral subjectivity). To help achieve this harmony between theoretical foundation and applicability, in this work we consider a framework built within the AHP that provides a mechanism for improving consistency based on a process of linearization developed by the authors. This provides the rigorous counterpart of our scheme. Its applicability hinges on the fact that it is a simple approach that can simultaneously deal with a wide number of applications. Finally, various case studies within the industrial field are presented that support the use of the linearization process to help bridge the gap between theoretical validity and applicability in MCDM.

Keywords: MCDM; AHP; consistency, linearization process

1. Introduction

In scientific problem-solving practice, especially in the case of complex problems (Hennissen et al. 2017, La Rocca et al. 2017), the classical separation between objective and subjective, quantifiable and qualitative, tangible and intangible, rational and emotional, etc., more than frequently does not occur. On the contrary, the neutrality of values demanded by theory-driven Science is an unrealistic hypothesis (Söderbaum 1999; Kaufmann 1999). This is especially clear in decision-making where subjective, not easily quantifiable, intangible, etc., aspects associated with human behaviour, which are the key players in decision-making processes, are especially present. To narrow this classic gap between theory and application, it is absolutely essential to incorporate the human factor into theoretical models (Dittrich, 2016), especially when facing high complexity problems (De Tombe 2001). It is imperative that the chosen problem-solving methodology combines the quantifiable, objective, tangible and rational of classical Science with the qualitative, subjective, intangible and emotional of human behaviour (Kunz 2015). Only in this way will it be possible to achieve an objective treatment of the subjectivity (Keeney 1992), so that an adequate rational treatment of the emotionality can be achieved. In this contribution we consider one of the most widespread multi-criteria decision-making techniques (Petruni et al. 2017, Kolahi et al. 2017, Szulecka and Monges Zalazar 2017, Aşchilean et al. 2017), the Analytic Hierarchy Process (AHP) (Saaty 1977, 2008). We consider that the application of AHP allows at clearly narrowing that classic gap between theory and practice. Within the AHP, as a mechanism to aid in the improvement of consistency, we present the linearization theory developed by the authors (Benítez et al. 2011a), which combines rigor and applicability at a time. Rigor is provided by the underlying mathematical content. The applicability derives from the simplicity and capability of the technique to be effectively applied to various decision problems. Consistency improvement is one of the key points of AHP, since the quality of the decision is closely related to the consistency of the judgments issued (Bulut 2012; Hillerman et al. 2017). As judgments are not necessarily consistent, something inherent to the human condition, tools are needed to improve consistency (Franek and Kresta 2014; Wang and Chen 2008). Improvement of consistency is the main objective of this contribution. Consistency is crucial in decision-making because it would be unwise to make decisions based on judgments that may seem to have been produced at random. When consistency for a matrix is not satisfactory, it is necessary to improve it. Finan & Hurley

(1997) stated that additional artificial manipulation to increase consistency will improve, on average, the reliability of the analysis. So, if consistency is unacceptable, it should be improved.

2. AHP basics: fundamentals and consistency

The AHP approach mainly consists of three stages, namely construction of the hierarchy of problem ingredients (i.e. objective, criteria and alternatives), calculation of the priorities of the elements, and aggregation of results to produce the final decision. Interactions between the elements are considered when building the structure of the problem. The elements are evaluated using Pairwise Comparisons (PCs), by asking experts or stakeholders involved in the decision-making problem about how much importance a criterion has when compared with another criterion of the same hierarchical level with respect to the interests or preferences of respondents. The candidate alternatives are also evaluated by PCs with respect to what is the higher degree of satisfaction for each criterion. Both kinds of related values can be determined by using various scales among which the most used one is the nine-point scale of Saaty (1977) where the range 1–9 moves from equal importance to extreme importance. Performing such a comparison yields an $n\times n$ matrix $A = (a_{ij})$, whose (positive) entries must adhere to two important properties, namely, $a_{ii} = 1$ (homogeneity) and $a_{ji} = 1/a_{ij}$ (reciprocity), i, $j = 1, \ldots, n$. The problem for matrix A becomes one of producing for the n elements, $E_1, \ldots, E_n$ (criteria or alternatives) under comparison, a set of numerical values $w_1, \ldots, w_n$ that reflect the priorities between pairs of compared elements according to the elicited judgments. If all judgments are completely consistent, the relations between weights w_i and judgments a_{ij} are simply given by $w_i/w_j = a_{ij}$ $(i, j = 1, \ldots, n)$, and the matrix A is said to be consistent. The following theorem provides equivalent conditions for a matrix A to be consistent. Firstly, we provide some notation. $M_{n,m}$ will hereinafter denote the set of $n\times m$ real matrices, and $M^+_{n,m}$ will denote the subset of positive matrices of $M_{n,m}$. It will be assumed that the elements of $\mathbb{R}^n$ are column vectors. For a given $A \in M_{n,m}$, let us write $[A]_{ij}$ the (i, j) entry of matrix A. The superscript ^{T} denotes the matrix transposition. The mapping $J : M^+_{n,m} \rightarrow M^+_{n,m}$ defined by $[J(A)]_{ij} = 1/[A]_{ij}$ will play an important role in the sequel.

Theorem 1 (Benítez et al. 2012a, Theorem 1). Let $A = (a_{ij}) \in M^+_{n,n}$. The following statements are equivalent.

(i) There exists $\mathbf{x} \in M^+_{n,1}$ such that $A = J(\mathbf{x})\mathbf{x}^T$.

(ii) There exists $\mathbf{w} = [w_1 \ldots w_n]^T \in M^+_{n,1}$ such that $a_{ij} = w_i/w_j$, for $i, j = 1, \ldots, n$.

(iii) $a_{ij}a_{ji} = 1$ and $a_{ij}a_{jk} = a_{ik}$ hold for all $i, j, k = 1, \ldots, n$.

For a consistent PC matrix, the leading eigenvalue and the principal (Perron) eigenvector provide information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector (Saaty 2008). As any consistent matrix has rank one (Benítez et al. 2012a), any of its normalized rows and, in particular, the normalized vector of the geometric means of the rows, also provides the priority vector. Taking into account the natural lack of consistency of human thinking, some degree of inconsistency is expected and, as a result, in general, A is not consistent. As shown in (Saaty 2003) the eigenvector is necessary for obtaining priorities. The hypothesis that the estimates of these values are small perturbations of the "right" values guarantees a small perturbation of the eigenvalues (see, e.g. (Stewart 2001)). For non-consistent matrices, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{max}\mathbf{w}$, where λ_{max} is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector. As a measurement of inconsistency, Saaty proposed using the so-called consistency index $CI = (\lambda_{max} - n)/(n - 1)$ and the consistency ratio $CR = CI/RI$, where RI is the so-called average consistency index (Saaty 2008). If $CR < 0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $CR < 0.1$.

3. Linearization: a technique to improve consistency

Several alternatives to improve consistency, mostly based on optimization, may be found in the literature. In this section, we firstly present the nonlinear nature of some of those methods. Then, we describe the linearization technique as an orthogonal projection mechanism over a certain vector space, and present a simple formula that implements this technique. Finally, we argue that the contribution of the expert is essential at all times.

3.1 Non-linear nature of optimization methods to improve consistency

Basically, optimization methods to improve consistency are based in Saaty's proposal (Saaty, 2003) based on perturbation theory to find the most inconsistent judgments in the matrix while adhering to some constraints. Thus, in general, slight modifications of the comparison matrix entries are sought, while maintaining the main properties of the comparison matrix, namely homogeneity, reciprocity and consistency. In (Aznar and Guijarro, 2008) it is proposed a goal programming method that uses relative deviations to force changes in the comparisons' values so that the target values differ as little as possible from the original values – while approximately taking homogeneity into account and preserving reciprocity and consistency. A slight modification of this method that reduces the number of decision variables and constraints is used in (Delgado-Galván et al. 2010). However, in (Benítez et al. 2012a) the authors provide an optimization process that has the important advantage of depending only on n decisional variables - the number of compared elements. The solution makes use of the Theorem 1 to solve the problem. Find a positive n-vector $\mathbf{x}$ such that

$$\min_{\mathbf{y}} \left\| A - J(\mathbf{y})\mathbf{y}^T \right\|_F = \left\| A - J(\mathbf{x})\mathbf{x}^T \right\|_F \text{ and } \left\| \mathbf{x} \right\|_1 = 1,$$

where $\mathbf{y}$ is a positive n-vector, $\|\cdot\|_F$ is the matrix Frobenius norm and $\|\cdot\|_1$ is the vector 1-norm. Note that $\|A\|_F = [\mathrm{tr}(A^T A)]^{1/2}$, where $\mathrm{tr}(\cdot)$ stands for the trace of a matrix, and the 1-norm for n-vectors is $\|y\|_1 = |y_1| + \ldots + |y_n|$.

To solve this optimization problem one may use, for instance, the Lagrangian multipliers. However, this is still a non-linear optimization problem. In the next subsection we present the main result of this contribution, namely a linearization technique that transforms the consistency improvement problem into a linear one.

3.2 Consistency through linearization

The linearization technique (Benítez et al. 2011a) provides the closest consistent matrix to a given non-consistent matrix by using an orthogonal projection on a certain linear space. This method provides a direct way of achieving consistency, in contrast with methods relying on non-linear optimization, which are iterative by nature.

The inspiration for the linearization methods comes from the following example.

Example Let us consider the PC matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}.$$

These four matrices are reciprocal (consistent, since they are 2×2) and correspond to four situations in which one must choose the best choice between two elements.

Using the Frobenius norm we have

$$\|A_1 - B_1\|_F = 1.118, \quad \|A_2 - B_2\|_F = 1.001.$$

This, somehow, shows that A_1 and B_1 resemble in a similar way as A_2 and B_2 do. This is not intuitive, since A_1 reflects the fact that both criteria are equally important, while B_1 gives double importance to the first over the second. In contrast, A_2 and B_2 show similar importance for both criteria. From an intuitive viewpoint the distance between A_1 and B_1 should be much higher than the distance between A_2 and B_2. Taking the example further, to allocate 100 euro between two competing options, the allocations obtained from these four matrices would be the ones given in Table 1.

Table 1: allocation for various PC matrices

Amount allocated to the…	A_1	A_2	B_1	B_2
…first option	50	66.3	88.9	90
…second option	50	33.3	11.1	10

We observe that the change from A_1 to B_1 allocations is much higher than from A_2 to B_2, as intuitively expected. So, just the Frobenius norm is not a good way to measure distances between matrices for this problem. However, by taking logarithms one can observe a more 'reasonable' jump between 1 and 2 than between 8 and 9, since log2 – log1 ≈ 0.693 and log8 – log9 ≈ 0.118. To conclude the example: we can conjecture that a new way to measure distances $d(A_1,B_1)$ between the pairwise comparison matrices A_1 and B_1 could be computed as:

$$\mathrm{d}(A_1, B_1) = \|\mathrm{L}(A_1) - \mathrm{L}(B_1)\|_F, \tag{1}$$

where L(·) is the matrix operator that associates the entries of a positive matrix with their logarithms, $[\mathrm{L}(X)]_{i,j} = \log([X]_{i,j})$. With this definition $\mathrm{d}(A_1,B_1) \approx 0.98$, while $\mathrm{d}(A_2,B_2) \approx 0.17$, which confirms the intuition that the distance between A_1 and B_1 should be much higher than the one between A_2 to B_2.

So, we keep on using the Frobenius norm, because of its simplicity. However, we will measure distances between PC matrices using (1).

Another advantage of using the map L is that we can use methods of linear algebra to improve consistency by solving an approximation problem in terms of the orthogonal projection of L(A) onto a linear subspace of $M_{n,n}$. To complete the details, let us define this subspace as $\mathcal{L}_n$ = {L(A) : A is a positive $n\times n$ consistent matrix}, which can be proved to be an (n – 1)-dimensional linear subspace of $M_{n,n}$. The complete process of getting consistency through linearization can be described by the following scheme (2):

$$A \xrightarrow{\mathrm{L}} \mathrm{L}(A) \xrightarrow{p_n} p_n(\mathrm{L}(A)) \xrightarrow{\mathrm{E}} A^c, \tag{2}$$

giving A^c, the closest consistent matrix to A; the operator E is defined for any matrix, X, by $[\mathrm{E}(X)]_{i,j} = \exp([X]_{i,j})$.

The first and third steps are trivial. So, only calculating $p_n(\mathrm{L}(A))$, the orthogonal projection of L(A) is needed.

The solution is guaranteed through standard linear algebra; see e.g. (Meyer 2001, p. 435). This projection is given by the formula in the following result.

Theorem 2 (Benítez et al. 2011a) Let $A \in M^{+}_{n,n}$. Then

$$p_n(\mathrm{L}(A)) = \frac{1}{2n}\sum_{i=1}^{n-1} \frac{\mathrm{tr}\left(\mathrm{L}(A)^T \phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|^2} \phi_n(\mathbf{y}_i) \tag{3}$$

is the closest matrix to L(A) in $\mathcal{L}_n$, where $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of the orthogonal complement to span$\{\mathbf{1}_n\}$, where $\mathbf{1}_n$ is the n-vector $[1,...,1]^T$, and $\phi_n(\mathbf{x})$ is the map that associates to a vector $\mathbf{x}$ the matrix with components $x_i - x_j$.

The following result shows that the calculations involved in the Fourier expansion given by (3) are straightforward.

Theorem 3 (Benítez et al. 2011a) Let $(Y_n)^{\infty}_{n=2}$ be the sequence of matrices defined as follows:

$$Y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Y_{n+1} = \begin{bmatrix} Y_n & 1_n \\ 0 & -n \end{bmatrix}, n \geq 2. \tag{4}$$

Then for every $n \geq 2$, the columns of Y_n are orthogonal and belong to the orthogonal complement of span$\{\mathbf{1}_n\}$.

Formulas (3) and (4) are extremely simple and require few operations. The implementation of these formulas in a conventional spreadsheets is really simple. However, matrix environments such as Matlab or Octave are deemed more appropriate (see straightforward implementations of (3) and (4) in Appendix A).

3.3 A simple formula for the projection

In (Benítez et al., 2013) the authors have shown that, for reciprocal matrices, this projection can be obtained with great simplicity by using the formula

$$p_n(L(A)) = \frac{1}{n}\left[(L(A)U_n) - (L(A)U_n)^T\right], \tag{5}$$

where $U_n = \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$.

Since this formula involves only sums, computational efficiency is guaranteed and integration in any AHP-based decision support system, including conventional spreadsheets is straightforward.

Example Let us consider the reciprocal matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1/3 & 1 & 2 \\ 1 & 1/2 & 1 \end{bmatrix}.$$

By using the Saaty's criterion of consistency we get that $CI/RI \simeq 0.35$. According to this criterion, the consistency of matrix A is not acceptable. We then modify matrix A to improve its consistency. We can apply (4) to get

$$p_3(L(A)) = \begin{bmatrix} 0 & 0.501 & 0.597 \\ -0.501 & 0 & 0.096 \\ -0.597 & -0.096 & 0 \end{bmatrix}.$$

Now the consistent matrix closest to A is

$$\mathrm{E}(p_3(L(A))) = \begin{bmatrix} 1 & 1.65 & 1.82 \\ 0.61 & 1 & 1.10 \\ 0.55 & 0.91 & 1 \end{bmatrix}.$$

However, maybe the experts can consider that this new matrix does not represent their opinions. For example $[A]_{1,2} = 3 > 1 = [A]_{1,3}$, while in the new matrix, the entry (1,2) is lower than the entry (1,3).

It is important to note that matrix $\mathrm{E}(p_n(\mathrm{L}(A)))$ should be never the last matrix—unless it reflects the thoughts of the expert. There must be a feedback between the expert judgments and the information conveyed by the matrices obtained by using the linearization method given by (3) or (4). We consider this aspect in the next section.

3.4 Balancing consistency and expert judgment

In light of the last example, we see that the synthetic consistency obtained through the linearization method must be in any case subjected to the final approval by the expert that issued the judgments. Therefore, after computing the closest consistent matrix given by the linearization method, it is necessary for the expert to be able to modify the new matrix. Following a feedback procedure, by repeating both steps, a matrix representing a reasonable trade-off between consistency and expert opinion will be eventually obtained.

Let us suppose that a reciprocal matrix A is obtained from a stakeholder judgment and the consistent matrix $A^c = \mathrm{E}(p_n(\mathrm{L}(A)))$ closest to A is calculated. Perhaps this actor does not completely agree that the entries in A^c fully represent his or her judgment. If the stakeholder decides to change, let us say, the entry a_{rs} in A^c, which compares criteria r and s (where $r \neq s$ and $1 \leq r, s \leq n$), another reciprocal, probably non-consistent, matrix B is obtained. The entries of B compared with the entries of A^c verify: $b_{rs} = \alpha a_{rs}$ and $b_{sr} = \alpha^{-1} a_{sr}$ for some $\alpha > 0$, and $b_{ij} = a_{ij}$ in the remaining entries.

Let us denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis of $\mathbf{R}^n$. The relationship between matrices $\mathrm{L}(A)$ and $\mathrm{L}(B)$ is

$$L(B) = L(A) + \log\alpha\left(\mathbf{e}_r\mathbf{e}_s^T - \mathbf{e}_s\mathbf{e}_r^T\right). \tag{6}$$

Using now the linearity of the projection one can state the following result (Benítez et al 2011b).

Theorem 4 Let $A \in M^+_{n,n}$ and let A^c be the consistent matrix closest to A. If B is defined by (6) and B^c is the consistent matrix closest to B, then

$$B^c = A^c \otimes \mathrm{E}\left(\frac{\log\alpha}{n}(\mathbf{e}_r - \mathbf{e}_s)\mathbf{1}_n^T - \mathbf{1}_n(\mathbf{e}_r - \mathbf{e}_s)^T\right). \tag{7}$$

$\otimes$ is the Hadamard (component-wise) matrix product.

4. Applications of the linearization scheme

Besides the obvious application in consistency improvement, the linearization methods may be extended to other usual applications in AHP. In this section we shortly present some of them.

4.1 Including (withdrawing) new (obsolete) elements

We consider the following situation. We have a reciprocal matrix, A, of size $n\times n$, and by virtue of the linearization process we get its closest consistent matrix A^c. Now, for any reason, a new element is introduced in the decision-making process, which is pairwise compared with the old elements. Is there a way to work on consistency considering the new judgments within the process without calculating everything from scratch? The following result (Benítez et al. 2012b) provides a positive solution.

Let B the $(n+1)\times(n+1)$-matrix obtained by enlarging A with the new judgments:

$$B = \begin{bmatrix} A & \mathbf{v} \\ J(\mathbf{v}') & 1 \end{bmatrix}, \tag{8}$$

where $\mathbf{v}$ is the n-vector of the new pairwise comparison between the old and the new elements.

Theorem 5 Let $A \in M^+_{n,n}$. For B defined as in (8), then $p_n(\mathrm{L}(B))$ is

$$\frac{1}{n+1}\begin{bmatrix} np_n(\mathrm{L}(A)) + \phi_n(\mathrm{L}(\mathbf{v})) & (\mathbf{1}_n\mathbf{1}_n^T + I_n)\mathrm{L}(\mathbf{v}) + p_n(\mathrm{L}(A))\mathbf{1}_n \\ * & 0 \end{bmatrix},$$

the block noted by * being determined by the fact that $p_n(\mathrm{L}(B))$ is skew-Hermitian.

In this Theorem the original matrix of pairwise comparisons, A, is used to obtain the new consistent matrix corresponding to the enlarged decision problem. Nevertheless, it is quite likely for the original matrix to have already been overridden by its associated consistent matrix. It is, thus, more natural to use the latter instead of the original matrix to build the consistent matrix for the enlarged problem. In (Benítez et al. 2012b) we show that identical results are obtained in both cases.

Withdrawing obsolete elements follows an inverse action of the previous one (Benítez et al. 2012b).

4.2 Clustering large pairwise comparison matrices

In highly complex problems, the number of elements to be compared may be very large. The linearization technique may be used to develop a merging technique so that some elements may be synthesized to produce a new comparison matrix that gathers elements into clusters, while maintaining the experience and the perception of the experts, and also the consistency, eventually, reducing the size of the problem, thus making it more manageable.

Let $A \in M^{+}_{n+m}$ be a consistent matrix and let us partition A as follows:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad A_1 \in M_n, A_4 \in M_m \,. \tag{9}$$

It is clear that both A_1 and A_4 are consistent. Now consider $M = \mathrm{L}(A)$ decomposed as follows:

$$M = \begin{bmatrix} M_1 & -M_2 \\ M_2^T & M_4 \end{bmatrix}, \quad M_1 \in M_n, M_3 \in M_m \,. \tag{10}$$

Theorem 6 Let $A \in M^{+}_{n+m}$ be a consistent matrix decomposed as in (9) with priority vector $[z_1, \ldots, z_{n+m}]^{\mathrm{T}}$ and $M = \mathrm{L}(A)$ be decomposed as in (10). Let N be produced by the clustering of the last m judgments of M and, finally, let us denote $\mathbf{w}_1 = [\log z_1 \ \ldots \ \log z_n]^{\mathrm{T}}$ and $s_2 = \log z_{n+1} + \ldots + \log z_{n+m}$. Then

$$N = \begin{bmatrix} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix}, \text{ where } \mathbf{v} = -\mathbf{w}_1 + \frac{s_2}{m}\mathbf{1}_n \,, \tag{11}$$

and the priority vector for $\mathrm{E}(N)$ is $\left[z_1, \cdots, z_n, \sqrt[m]{z_{n+1} \cdots z_{n+m}}\right]^T$.

If the matrix is close to being consistent (e.g., its consistency is acceptable according to Saaty's criterion), then we can apply this Theorem (approximately) to obtain the collapsed matrix and its priority vector.

4.3 Completing information

It is natural that some of the actors involved are not familiar enough with all the issues to make appropriate comparison judgments. As a result, it is difficult to gather complete information about the preferences of such a decision maker at a given moment. In (Benítez et al. 2014, 2015, 2017) we address this scenario. Using the linearization theory we characterize when an incomplete, positive, and reciprocal matrix can be completed to become a consistent matrix in terms of graph theory, and provide matrix completion mechanisms.

In this contribution we just provide the simplest results about the completion of an incomplete PC matrix. Let us suppose that the k unknown entries of A above its diagonal are in the positions (i_r, j_r), $r = 1, \ldots, k$, with $1 \le i_1, j_1, \ldots, i_k, j_k \le n$ and $i_r < j_r$, $r = 1, \ldots, k$. Of course, their symmetric positions are also unknown. First, observe that $\mathrm{L}(A)$ can be written as

$$\mathrm{L}(A) = B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} \,, \tag{12}$$

where B_0 is the skew-Hermitian matrix with entries the logarithms of the known entries of A and 0's elsewhere, λ_r are unknown numbers, and $B_{i_r j_r}$ is the matrix such that its (i_r, j_r)-entry equals 1, its (j_r, i_r)-entry equals -1, and all the other entries vanish.

Observe that B_0 can be written as $B_0 = \sum_{i<j} \rho_{ij} B_{ij}$, for adequate values ρ_{ij}.

Then, the problem, using the notation used in (12), may be stated as follows.

Problem Let $B_0 \in M_{n,n}$ be a skew-Hermitian matrix and $1 \le i_1, j_1, \ldots, i_k, j_k \le n$ be indices with $i_r < j_r$, $r = 1, \ldots, k$. Find $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$ and $\mu_1, \ldots, \mu_{n-1} \in \mathbf{R}$ such that

$$\left\| B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu_\sigma \phi_n(y_s) \right\|_F \le \left\| B_0 + \sum_{r=1}^{k} \lambda'_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu'_\sigma \phi_n(y_s) \right\|_F$$

For all $\lambda'_1, \ldots, \lambda'_k$, $\mu'_1, \ldots, \mu'_{n-1}$ real numbers.

The following theorem gives the values of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)^{\mathrm{T}}$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n-1})^{\mathrm{T}}$ sought.

Theorem 8 Let B_0 the skew-Hermitian matrix defined in (12) and $1 \le i_1, j_1, \ldots, i_k, j_k \le n$ with $i_r < j_r$, $r = 1, \ldots, k$. Assume that the (i_r, j_r)-entry of B_0 is zero for $r = 1, \ldots, k$. The solution of the previous problem satisfies

$$\boldsymbol{\lambda} = S\boldsymbol{\mu}, \quad \left(D - \frac{1}{n} S^T S\right)\boldsymbol{\mu} = \mathbf{b} \,, \tag{13}$$

where S is the $k \times (n-1)$ matrix whose (r,s) entry is $\mathbf{d}_{i_r j_r}^T \mathbf{y}_s$, with $\mathbf{d}_{ij} = \mathbf{e}_j - \mathbf{e}_i$, D is the diagonal $(n-1) \times (n-1)$ matrix whose (s,s) element is $\| y_s \|^2$, and $\mathbf{b} = [\mathbf{w}^{\mathrm{T}} \mathbf{y}_1, \ldots, \mathbf{w}^{\mathrm{T}} \mathbf{y}_{n-1}]^{\mathrm{T}}$, being $\mathbf{w} = \frac{1}{n} \sum_{i<j} \rho_{ij} \mathbf{d}_{ij}$.

Despite the apparent complexity of this result, the calculations are straightforward and can be easily codified in spreadsheets, or in MatLab as in Appendix B.

5. Case studies

Given the possibility of successfully applying AHP in many fields and problems and integrating it with other techniques (Ho 2008), a plethora of applications is discussed in the literature. Vaidya and Kumar (2006) present a wide literature review and, after revising a sample of 150 papers on AHP, provide a wide number of AHP applications. In this section we concisely present two applications within the authors' field of expertise.

5.1 Deciding a leak control policy in water supply

The minimization of water loss poses a great challenge to water supply managers. Great sums of money are devoted annually to this aim worldwide. We consider here two management alternatives for leakage control. Active leakage control (ALC) involves taking actions in supply systems to identify and repair not reported leaks. Passive leakage control (PLC) boils down to just repairing reported or evident leaks (Farley and Trow 2003). Various criteria, including tangible and intangible factors and qualitative factors, are used to decide on the alternatives.

To illustrate the application of the methodology developed, we consider a set of seven criteria:

C1: cost of planning development and implementation;
C2: damage to property and other service networks;
C3: effects (cost or compensations) of supply disruptions;
C4: inconveniences caused by closed or restricted streets;
C5: water extractions (benefits for aquifers, rivers, etc.);
C6: environmental and recreational impacts;
C7: CO_2 emissions (related to energy used in pumping).

We use here the point of view of the management department of a real water supply company. Upon criteria reordering and evaluation using the 9-point Saaty scale, the matrix A in Table A1 (Appendix C) was produced. The Perron eigenvector (the priority vector), normalized to sum 1, is $\mathbf{Z} = [0.42, 0.23, 0.13, 0.09, 0.07, 0.03, 0.03]^T$. This matrix is positive, homogeneous and reciprocal, yet inconsistent. Since $\lambda_{max} \approx 10.56$, $CI \approx 0.5934$, and $CR \approx 43.96\%$, consistency of this matrix is inadmissible.

The process described in Section 3.4, gives the final consensus between consistency and expertise (Table A2 in Appendix C). The Perron eigenvector (the priority vector), normalized to sum 1, is $\mathbf{Z} = [0.25, 0.20, 0.21, 0.14, 0.12, 0.06, 0.02]^T$. Thus, greater importance is placed on criterion C_1 closely followed by C_3 and C_2.

We omit in this document, the final aggregation process, which would normally take place and make use of a comparison of alternatives for each criterion, since our main objective here is to exemplify the trade-off between synthetic consistency, using (7), and expert judgment.

5.2 Location of shelves for materials handling

The AHP technique is herein applied to deal with a problem of industrial layout reorganisation with incomplete information. The objective is the selection of the best arrangement of shelves, *i.e.* layout proposal, into the storage of a firm to store assets, which are cardboards and pallets of finished products. Three layout proposals (LP_1, LP_2, LP_3) with various placements of shelves are evaluated on the basis of five criteria (C_1, C_2, C_3, C_4, C_5), namely: safety & security, cost, innovation, transport and placement. All the alternatives were evaluated with respect to each considered criterion; consistency was verified, and local priorities were achieved *via* the power method. Moreover, a team composed by three decision makers (D_1, D_2, D_3) was involved in evaluating the vector of criteria weights. The experts decided not to express some judgments when they felt not sufficiently confident about them. In particular, expert D_1 did not express just one PC, whereas experts D_2 and D_3 did not formulate two PCs. As a result, three incomplete matrices needed to be filled in before aggregating judgments in a single matrix. The unique set of λ's providing the completions are in Table 2.

Table 2. λ values calculated using (13)

	D_1	D_2	D_3
λ	-0.24731	-0.19822	-0.81944
		-0.37150	-1.54203

The achieved completions of matrices are shown in Tables A3-A5 (Appendix C). The new values are in bold.

The three matrices were blended into one aggregated matrix by means of the Aggregation of Individual Judgments (AIJ) technique. In this way, the decision group could be considered as a unique "new individual". The priority vector for this consistently completed matrix shows the main importance of safety & security:

$\mathbf{w} = [0.384\ 0.0843\ 0.2061\ 0.1482\ 0.1773]^T$.

Upon having calculated the priority vector via the power method, results were aggregated through the distributive method, and the final ranking of layout proposals was achieved: alternative LP1 got the best score, 0.5442; then alternative LP3 got the second, 0.2564; and, finally, alternative LP2 had the lower priority with a coefficient or 0.1993.

Conclusions

Methodological approaches able to bridge the gap between theory and practice need to adequately combine traditional theory-driven science objectivity and human behavior subjectivity. In this contribution, built upon the AHP context, a methodology which has broadly proven to conciliate theory and practice, we present the linearization scheme. It is a rigorous framework, which provides a mechanism of consistency improvement, a crucial aspect in decision-making, and which, at the same time, exhibits contrasted applicability. Various industrial case studies help us justify the theoretical validity and applicability of the linearization method within the MCDM field.

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Appendix A. Matlab or Octave codes for (3) and (4)

```
function y = y(n)
% This function calculates matrices Y in (4)
y = zeros(n,n-1);
for k=1:n-1
       y(1:k,k)=ones(k,1);
       y(k+1,k)=-k;
end

function matrix = matrix(A)
% Calculates sought consistent matrix in (3)
B = log(A);
[n m] = size(A);
Y = y(n);
X = zeros(size(A));
for i = 1:n-1
       phiy = Y(:,i)*ones(1,n)-
ones(n,1)*Y(:,i)';
       factor = trace(B'*phiy)/(i+i^2);
       X = X + factor*phiy;
end
X = X/(2*n);
matrix = exp(X);
```

Appendix B. Matlab or Octave codes for (13)

The following `m-file` follows the notation in (13), except for matrix S, which is stored as `alpha`.

```
function [lambda mu] = th4(A, P)
% A is the matrix to be completed
% (if we do not know a {ij}, then A(i,j)=1.
% P is a (0, 1)-matrix such that
% if we do not know a {ij}, then P(i,j)=1.
% if we know a {ij}, then P(i,j)=0.
% Use: [lambda mu] = th4(A, P)
[n, m] = size(A);
B = log(A);
auxP = triu(P);
[noi noj] = find(auxP==1);
auxP = triu(P+ones(n,n),1);
[sii sij] = find(auxP==1);
kn = length(noi); ks = length(sii);
Y = y(n);
D = diag(ones(1,n)*Y.^2);
I = eye(n);
w = zeros(n,1);
alpha = zeros(kn,n-1);
for r = 1:kn
for s = 1:n-1
i = noi(r); j = noj(r);
alpha(r,s) = (I(j,:)-I(i,:))*Y(:,s);
end
end
for index = 1:ks
i = sii(index); j = sij(index);
w = w+B(i,j)*(I(:,j)-I(:,i));
end
w = w/n;
b = Y'*w;
mu = (D - alpha'*alpha/n)\b;
lambda = alpha*mu;
```

This file uses another `m-file`, `y.m` which computes the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}$. We include this here for the sake of the completeness.

```
function y = y (n)
y = zeros(n,n-1);
for k = 1:n-1
y (1:k,k) = ones (k,1);
y (k + 1,k) = -k;
end
```

Appendix C. PCMs for the case studies in Section 5

Table A1: Initial matrix of comparison of criteria

	C1	C2	C3	C4	C5	C6	C7
C1	1	7	9	5	7	5	3
C2	1/7	1	5	9	5	7	5
C3	1/9	1/5	1	7	3	7	3
C4	1/5	1/9	1/7	1	7	5	5
C5	1/7	1/5	1/3	1/7	1	9	7
C6	1/5	1/7	1/7	1/5	1/9	1	5
C7	1/3	1/5	1/3	1/5	1/7	1/5	1

Table A2: Consistently completed matrix

	C1	C2	C3	C4	C5	C6	C7
C1	1	1.28	1.17	1.77	2.12	4.49	9.89
C2	0.78	1	0.92	1.39	1.66	3.52	7.74
C3	0.85	1.09	1	1.51	1.81	3.83	8.42
C4	0.57	0.72	0.66	1	1.20	2.54	5.59
C5	0.47	0.60	0.55	0.83	1	2.16	4.66
C6	0.22	0.28	0.26	0.39	0.47	1	2.20
C7	0.10	0.13	0.12	0.18	0.22	0.45	1

Table A3: Completed matrix D_1

D_1	C1	C2	C3	C4	C5
C1	1	7	1	4	5
C2	1/7	1	1/3	1/3	**0.78090**
C3	1	3	1	4	3
C4	1/4	3	1/4	1	2
C5	1/5	**1.28058**	1/3	1/2	1

Table A4: Completed matrix D_2

D_2	C1	C2	C3	C4	C5
C1	1	5	3	3	2
C2	1/5	1	**0.82019**	2	**0.68980**
C3	1/3	**1.21922**	1	3	1/2
C4	1/3	1/2	1/3	1	1
C5	1/2	**1.44991**	2	1	1

Table A5: Completed matrix D_3

D_3	C1	C2	C3	C4	C5
C1	1	5	1	2	1
C2	1/5	1	**0.44068**	1/3	**0.21394**
C3	1	**2.26923**	1	1/2	1/3
C4	1/2	3	2	1	1
C5	1	**4.67409**	3	1	1